Documento de Trabajo
8 6 0 4

THE ROLE OF ADJUSTMENT COSTS IN INTEREST RATE DETERMINATION

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ABSTRACT.

We study in this paper the equilibrium influence of adjustment costs of capital on interest rates determination. Considering endogenous interest rates in optimal capital accumulation models introduces nonlinearities which together with expectations of future variables make the model hard to analyze. We use here a solution method that has recently been proposed in the literature, to show that the model is able to reproduce some of the correlations among output, consumption, interest rates and capital that can be observed in actual time series data.

I. INTRODUCTION.

The role of adjustment costs in the determination of the optimal capital accumulation path has been emphasized in the economic literature (see Treadway [9] and [10], Gould [1], Lucas [5] and Mortenson [6]). In a general equilibrium framework, fluctuations in the optimal stock of capital will affect as well as be influenced by interest rates. These, in turn, will be intertemporally correlated with output and consumption. Such a model could therefore be used to explain and interpret a number of dynamic statistical properties of real economies that have been observed in empirical work (see Litterman and Weiss [4], among others).

Interest rates have however been assumed in previous related work to be constant, or moving in an exogenously given deterministic fashion. A practical reason for such an assumption is that to make interest rates endogenous introduces nonlinearities in the model. Under uncertainty, the simultaneous presence of nonlinearities and expectations of future variables will preclude obtaining a closed form solution to the model, making impossible to analytically characterize the model's properties. An alternative is to generate sample realizations for the vector stochastic process of variables in the economy. We could then obtain some time series statistics: autocorrelation functions, cross-correlation functions for pairs of variables, impulse responses of the system to innovations in any one variable, decompositions of variance of forecast errors, and others. Functions like these have been used in previous empirical research to characterize the stochastic properties of univariate time series data as well as the intertemporal, dynamic interrelations between sets of variables. Therefore, the statistics obtained from the model could be compared with similar ones obtained from actual time series data, and small distances between these functions be used as criteria for goodness of fit of the model.

Unfortunately, the same analytical difficulties we have mentioned above make this data generation process a non trivial task. We utilize here a method suggested in Sims [8] and already utilized in Novales [7], to obtain explicit equilibrium solutions to nonlinear rational expectations models. The method does not need a closed form solution to the model, but requires some analysis so as to guarantee stability of the obtained solution.

The economy we consider is described in section 2, whereas in section 3 we propose the application of the solution method to this particular model. Section 3 contains a discussion of the calibration of the model, while in section 4 we present the numerical results that summarize the dynamic properties of the model. The paper closes with some conclusions and suggestions for further work.

II. THE ECONOMY.

Let us consider a single commodity economy with a continuum of identical agents. The commodity can either be consumed or used as capital. The stock of capital at time t is an input in the production of time $t+1$ resources, according to the production function:

$$(2.1) \quad Y_{t+1} = f(K_t, \epsilon_{t+1}) = \gamma - \frac{\theta}{2}\cdot(K_t-\alpha)^2 + \epsilon_{t+1}$$

where γ is a positive constant with $\gamma > \max\left\{0 \; ; \; \theta\cdot\alpha^2/2-\inf(\text{support } \epsilon_t)\right\}$, and ϵ_t is a stochastic process. The amunt produced each period is therefore a random function of the previous period stock of capital.

At each time t, the cost of varying the stock of capital is given by:

$$(2.2) \quad \Psi(K_t, K_{t-1}) = \omega/2\cdot(K_t-K_{t-1})^2$$

This quadratic specification for the adjustment costs of capital has been very popular in previous research (see references in the introduction). The convenience and interpretation of this assumption have been discussed in these references and will not be repeated here.

We assume the following single period utility function for the typical consumer:

$$(2.3) \quad U_t(C_t, C_{t-1}) = C_t - \frac{n}{2}\cdot C_t^2 - \frac{b}{2}\cdot(C_t-C_{t-1})^2$$

which is not time separable, for it makes the utility of current consumption a function of last period's consumption. From the functional form in (2.3), we can interpret the nonseparability as reflecting a distaste for rapid changes in the consumption values.

Consumers behave as price takers and solve the problem:

$$(2.4) \quad \max_{\{C_t,K_t\}} \quad E_0 V(\{C_s\}_{s=-1}^{\infty}) = E_0 \sum_{t=0}^{\infty} \beta^t \cdot U_t(C_t, C_{t-1})$$

subject to :

$$(2.5) \quad C_t + K_t - K_{t-1} = f(K_{t-1}, \epsilon_t) - \Psi(K_t, K_{t-1})$$

together with (2.1)-(2.3) and C_t, $K_t \geq 0$ for all $t \geq 0$. Since there are random shocks to technology, this becomes an optimization problem under uncertainty. The information set Ω_t based on which the conditional expectation E_t is calculated is assumed to contain: $\{Y_{t-s}, C_{t-s-1}, K_{t-s-1}, s \geq 0\}$. In what follows, we reserve the notation U_t for the period t utility, and denote by V the lifetime discounted aggregate utility.

A change in consumption at time t changes not only the utility of the current period, but next period's utility as well, due to the nonseparability of preferences. The marginal lifetime utility of current consumption then is:

$$\partial V/\partial C_t = \beta^t \cdot (1 - nC_t - b \cdot (C_t - C_{t-1})) + \beta^{t+1} \cdot b \cdot (C_{t+1} - C_t)$$

which can be seen to be a random variable to be realized at time $t+1$. It is convenient to introduce the process of discounted marginal utility of current consumption:

$$(2.6) \quad W_t = \beta^{-t} \cdot \partial V/\partial C_t = 1 + \beta \cdot b \cdot C_{t+1} - (n + b + b \cdot \beta) \cdot C_t + b \cdot C_{t-1}$$

which again, is a random variable at time t .

It is shown in the appendix that the following is a necessary condition for optimality of an interior solution:

$$(2.7) \quad [1 + \omega \cdot (K_t - K_{t-1})] \cdot E_t W_t = \beta \cdot E_t [W_{t+1} \cdot (1 + \omega \cdot (K_{t+1} - K_t) - \theta \cdot (K_t - \alpha))]$$

To interpret this condition, let us define the <u>net revenue function at</u>

time t on last period's capital stock as the gross output produced at time t minus the adjustment costs of capital between t-1 and t :

$$R_t(K_{t-1},K_t) = \gamma + K_{t-1} - \frac{\theta}{2}\cdot(K_{t-1}-\alpha)^2 + \epsilon_t - \frac{\omega}{2}\cdot(K_t-K_{t-1})^2$$

which can be seen to depend on the current stock of capital as well. Consider the partial derivatives:

$$\partial R_t/\partial K_t = -\omega\cdot(K_t-K_{t-1}) \ .$$

(2.8)

$$\frac{\partial R_{t+1}}{\partial K_t} = 1-\theta\cdot(K_t-\alpha)+\omega\cdot(K_{t+1}-K_t)$$

If the current stock of capital is above last period's K_{t-1} and we decrease K_t by one unit, the return at time t on K_{t-1} then increases. The same result is obtained when K_t is below K_{t-1} and we increase K_t. In this economy, an increase of a unit in the stock of capital requires of a decrease in consumption of one unit plus the marginal cost of adjusting the stock of capital. That may add to more or less than a decrease of one unit of commodity in consumption, depending on whether the difference K_t-K_{t-1} is positive or negative . From (2.8), the total change in current consumption needed to increase the stock of capital by one unit is therefore given by: $1-\partial R_t/\partial K_t$.

The additional unit of capital at time t has two real effects at time t+1 : a) it increases production (that is, output at time t+1) by the marginal product: $1-\theta\cdot(K_t-\alpha)$, and b) it contributes to the adjustment costs to be paid at t+1, since the stock of capital is now one unit larger. The aggregate effect is therefore given by $\partial R_{t+1}/\partial K_t$. Consumption at time t+1 can be increased by the net amount of these two effects. The implied utility gain at t+1 should exactly compensate the utility loss from having decreased consumption at time t to gain one additional unit of capital. But that is exactly the message in (2.7), which can be written:

$$E_t\left(1-\frac{\partial R_t}{\partial K_t}\right)\cdot W_t \quad = \beta\cdot E_t\left(\frac{\partial R_{t+1}}{\partial K_t}\cdot W_{t+1}\right) \tag{2.9}$$

In equilibrium, the representative agent is indifferent between his allocation and a movement along his feasible set that involves giving up enough current consumption so as to increase the stock of capital by one unit and consume tomorrow the output increment produced by this change.

Recently, different models have obtained and tested implications of the type: "The marginal utility of consumption behaves as a first order Markov process" (see Hall [2], for example). The model in this paper shows that with adjustment costs of capital the condition is somewhat different. If we denote:

$$\zeta_t = \beta\cdot\frac{\partial R_t}{\partial K_{t-1}}\Bigg/\left(1-\frac{\partial R_{t-1}}{\partial K_{t-1}}\right)$$

and define $\Phi_t = \Pi_{s=1}^{t}\zeta_s$, then, multiplying through (2.7) by Φ_t we get:

$$E_t(\Phi_t\cdot W_t) = E_t(\Phi_{t+1}\cdot W_{t+1})$$

With a time separable utility of consumption, the marginal utility discounted by the random factor Φ_t is a martingale process. When the utility function is not time separable, the discounted marginal utility $\Phi_t\cdot W_t$ then satisfies a condition which is weaker than a martingale: the current predictions of any two consecutive future values are the same. (notice that with our specified utility function, $\Phi_t\cdot W_t$ is Ω_{t+1}-measurable and $\Phi_{t+1}\cdot W_{t+1}$ is Ω_{t+2}-measurable) .

This result reduces to Hall's under his set of assumptions: Time separability of preferences implies that $E_t(\Phi_t W_t) = \Phi_t\cdot W_t$. Furthermore, without costs of adjustment, then Φ_{t+1}/Φ_t is equal to the discount factor β times the marginal productivity of capital, equal to one plus the real rate of interest. Therefore, under those conditions, we have:

$$E_t W_{t+1} = \frac{1}{\beta} \cdot \frac{1}{1+r_t} \cdot W_t \tag{2.10}$$

where W_t is the marginal utility of consumption, which is Hall's result. All these are different degrees of generalization of Hall's result about the behavior of consumption as a first order Markov process when the utility function is time separable and approximately quadratic in a neighborhood of the equilibrium steady state value of consumption, and the interest rate is constant.

One way to introduce interest rates in the model is by defining the real rate at each time t to be equal to the marginal rate of time preference:

$$1 + r_t = \frac{E_t W_t}{\beta \cdot E_t W_{t+1}} \tag{2.11}$$

This condition arises from the utility maximization problem the consumer solves in the case when there are some investment opportunities (this is shown in appendix 2). Under uncertainty, the equilibrium rates of return on all assets in the economy would be equal to the gross real rate of interest. From the way uncertainty enters our model, we get here a weaker condition:

$$E_t \left[\frac{\partial R_{t+1}(K_t, K_{t+1})/\partial K_t}{1 - \partial R_t(K_{t-1}, K_t)/\partial K_t} \cdot \beta \frac{E_t W_{t+1}}{E_t W_t} \right] = 1 \tag{2.12}$$

which is just a rewriting of (2.9), and shows that in the conditional expectation sense, the equilibrium return on capital is equal to the rate of time preference. Unfortunately, we cannot conclude that the two rates are the same (see appendix 2).

III. SOLUTION OF THE MODEL.

The model in the previous section has been shown to produce interesting dynamic relations among consumption, capital, interest rates and output. The joint presence of nonlinearities and expectations of future variables in the equilibrium conditions prevent us from obtaining the closed form solution that would be needed to analytically characterize the model's properties regarding the interrelationships among these variables. An alternative way to analyze the model could be to generate equilibrium time series realizations that would be used to compute statistics (autocorrelation functions, cross-correlation functions, impulse response functions), that would summarize the dynamics of the economy under study. Unfortunately, this alternative approach is far from trivial, due to the presence of the conditional expectations in the equilibrium conditions.

The equilibrium in the economy described in the previous section is characterized by the set of optimality conditions:

$$(3.1)\qquad [1+\omega\cdot\Delta K_t]\cdot E_t W_t = \beta\cdot E_t[W_{t+1}\cdot\{1+\omega\cdot\Delta K_{t+1}-\theta\cdot(K_t-\alpha)\}]$$

$$(3.2)\qquad 1+r_t = \frac{E_t W_t}{\beta\cdot E_t W_{t+1}}$$

the technology:

$$(3.3)\qquad Y_t = \gamma - \frac{\theta}{2}\cdot(K_{t-1}-\alpha)^2 + \varepsilon_t$$

and the budget constraint:

$$(3.4)\qquad C_t + K_t - K_{t-1} + \omega/2\cdot(K_t - K_{t-1})^2 = Y_t$$

together with the expression for the marginal utility of consumption:

(3.5) $W_t = 1 + a_o \cdot C_{t+1} + a_1 \cdot C_t + a_2 \cdot C_{t-1}$

with: $a_o = \beta \cdot b$, $a_1 = -(n+b+\beta \cdot b)$; $a_2 = b$,so that W_t is not realized until time $t+1$.

The equilibrium conditions (3.1)-(3.5) are five equations in $\{W_t, C_t, K_t, Y_t, r_t, \epsilon_t\}_{t=o}^{\infty}$. Actually, we are not one equation short, for we have not specified yet a stochastic representation for the technology shock ϵ_t which could be used to obtain realizations for it. Equivalently, this means that the model needs of an additional condition to be closed. The difficulties we have mentioned suggest that arbitrary ways of closing the model will not in general allow us to solve the model. In particular, assuming that ϵ_t follows a univariate ARIMA representation will let us generate realizations for ϵ_t , but will not allow us to use these realizations in (3.1)-(3.5) to generate data for all the other variables, again due to the simultaneous presence of conditional expectations and nonlinearities.

In order to follow that approach, the expectations in (3.1)-(3.4) raise an important difficulty, because that increases the number of variables to solve for, given that we then have not only realized current and past values of variables, but the values taken by the conditional expectations of future variables as well.

We close the model by considering the process:

(3.6) $$Z_{t+1} = W_t \cdot (1+\omega \cdot \Delta K_t)$$

The reason to date the process Z_t as in (3.6) is that with the assumed utility function (2.5), the discounted marginal utility is a variable to be realized at time $t+1$. The process Z_t can be used to write the optimality condition (3.1) as:

(3.7) $$\beta^{-1} \cdot E_t Z_{t+1} = E_t Z_{t+2} - \theta \cdot (K_t - \alpha) \cdot E_t W_{t+1}$$

Assumption 1.- Z_t is a stationary process that admits a first order autoregressive representation:

(3.8) $$Z_{t+1} = A \cdot Z_t + \mu + \nu_{t+1}$$

where ν_t is i., i.d., $N(0,\sigma_\nu^2)$ and $|A| < 1$.

This amounts to assuming Granger causal priority of Z_t with respect to all other variables in the economy. However, the Granger causal priority concept, as we usually think of it applies to linear models, and it is unclear what the assumption made above implies about the Granger causal properties of consumption and capital, the two components in the process Z_t . One would guess that the Granger causal priority of Z_t can be compatible with almost any possible causal ordering among C_t, K_t and all the other variables in the economy.

Stability Analysis:

Closing the model must be done in a way that achieves stability of the implied solution. We can analyze the stability properties of the model by constructing a linear approximation around its steady state. Equations (2.5) and (2.7) can be approximated around the steady state values (W^*, K^*, ϵ^*) by:

$$\begin{array}{c} (2.5) \\ \\ (2.7) \end{array} \begin{bmatrix} -(1-L) & W^* \cdot \left[\omega \cdot (1-L) \cdot (L-\beta) + \beta \cdot \theta \cdot L\right] & 0 \\ \\ Q^{-1}(L) & (1-L) \cdot (1-\omega \cdot K^*) + \theta \cdot (K^* - \alpha) \cdot L & -1 \end{bmatrix}$$

where $Q(.)$ is the lag polynomial (3.5) that defines W as a function of current, future and past consumption values. Suppose we now complete the system with a third row of the form: [E F 0], in which the last entry is taken to be zero because it simplifies the computations of the eigenvalues of the resulting matrix. It is not hard to show that if E is zero, the resulting matrix has then a determinant with a unit root. On the other hand, if F is chosen to be zero, then the determinant is a quadratic, with at least one of the roots inside the unit circle, and consequently, the system is not stable.

Therefore, we need to close the system with an assumption on the stochastic distribution of a function of both, W_t and K_t. Suppose that we assumw the autoregression: $B(L)\cdot\{W_t\cdot[1+\omega\cdot(K_t-K_{t-1})]\} = \nu_{t+1}$. The third row of the matrix then becomes:

$$\begin{bmatrix} B(L) & B(L)\cdot W^*\cdot\omega\cdot(1-L) & 0 \end{bmatrix}$$

and the lag polynomial determinant has as roots those of $B(L)$ together with:

$$L = \frac{-\omega\cdot(1-\beta)}{-\omega\cdot(1-\beta)+\beta\cdot\theta}$$

which needs of the relation: $\omega > \theta\cdot\beta/(1-\beta)$. This inequality is of course a constraint on the chosen parameter values when numerically solving the model. The solution will be stable so long as this condition among the parameter values is satisfied and the autoregression for Z_t is chosen to be stationary. Our equation (3.8) is just one such stationary autoregression.

Recursive solution of the model:

With assumption 1, equation (3.8) then becomes:

$$\beta^{-1}\cdot(AZ_t+\mu) = A^2\cdot Z_t + (A+1)\cdot\mu - \theta\cdot(K_t-\alpha)\cdot E_tW_{t+1}$$

which gives us the conditional expectation formula:

$$E_tW_{t+1} = \frac{A(A-\beta^{-1})Z_t-(\beta^{-1}-A-1)\cdot\mu}{\theta\cdot(K_t-\alpha)} \tag{3.9}$$

and suggests that:

$$W_{t+1} = \frac{A(\beta^{-1}-A)\cdot Z_t+(\beta^{-1}-A-1)\cdot\mu}{-\theta\cdot(K_t-\alpha)} + \eta_{t+2} \tag{3.10}$$

with $E_t\eta_{t+2}=0$. Again, the date in η_{t+2} comes about because W_{t+1} is realized at time t+2 . As a consequence of (3.10) , η_t is a first order moving average random shock:

$$\eta_t = \xi_t - \rho \cdot \xi_{t+1}, \qquad \text{where } \xi_t \text{ is a white noise.} \tag{3.11}$$

Equations (3.6) and (3.10) together give us:

$$K_t = K_{t-1} + \frac{1}{\omega} \cdot \left(\frac{Z_{t+1}}{\dfrac{A(\beta^{-1}-A)Z_{t-1}+(\beta^{-1}-A-1)\cdot\mu}{-\theta\cdot(K_{t-1}-\alpha)} + \eta_{t+1}} - 1 \right) \tag{3.12}$$

which is a recursive formula to determine the equilibrium path of capital accumulation as a function of the process Z_t . The equilibrium value of the interest rate is given by:

$$r_t = \frac{1}{\beta} \cdot \frac{E_tW_t}{E_tW_{t+1}} - 1 = \frac{\dfrac{A(\beta^{-1}-A)\cdot Z_{t-1}+(\beta^{-1}-A-1)\cdot\mu}{-\theta\cdot(K_{t-1}-\alpha)} + E_t\eta_{t+1}}{\beta \cdot \dfrac{A(\beta^{-1}-A)\cdot Z_t+(\beta^{-1}-A-1)\cdot\mu}{-\theta\cdot(K_t-\alpha)}} - 1 \tag{3.13}$$

Starting from initial conditions (Z_0,K_0,ξ_0) and a parameter vector $(A,\mu,\beta,\omega,\alpha,\theta,\sigma_\nu^2,\sigma_\epsilon^2)$, and drawing a random realization for the vector process (ν_t,ξ_t), we can then use (3.8) and (3.11) to generate time series for Z_t and η_t, and them together with K_0 in (3.12) to generate a time series for K_t . After that, (3.10) can be used to get a time series for W_t . The final step is to recover the equilibrium path of consumption from (3.5), the expression for the marginal utility of consumption W_t , which can be written:

$$W_t = 1+\beta.b.Q(L).C_{t+1}$$

with:

$$Q(L) = 1 - \frac{(n+b+\beta.b)}{\beta.b}.L + \frac{1}{\beta}.L^2$$

and therefore, we have for each time t the identity:

$$(3.14) \qquad Q(L).C_{t+1} = \frac{W_t-1}{\beta.b}$$

Let λ_1 and λ_2 denote the two roots of the lag polynomial $Q(L)$. They can be obtained from:

$$\lambda_1 + \lambda_2 = \frac{n+b+\beta.b}{\beta.b}$$

$$\lambda_1.\lambda_2 = 1/\beta$$

Given a value for the discount factor β, then a choice of λ_1 determines the value of λ_2 and finally, a value of n gives us the value of b by:

$$b = n / [\beta(\lambda_1+\lambda_2-1)-1]$$

Let us assume that the roots are chosen so that $|\lambda_1|>1$, $|\lambda_2|<1$. We can then expand (3.14) forward:

$$(3.15) \qquad C_{t+1} = \lambda_2.C_t + [b.\beta.(\lambda_1-1)]^{-1} - [\beta.b]^{-1} . \Sigma_{i=1}^{\infty}(\beta\lambda_2)^i.W_{t+i}$$

We want to use (3.15), together with a realization of the process of marginal utility W_t to compute the corresponding realization for consumption. Clearly, in order to do so, we will have to truncate the infinite sum in (3.15) at some $t+T$, and use the approximation:

$$(3.16) \quad C_{t+1} = \lambda_2 . C_t + \frac{1}{\beta . b} . \left(\frac{1}{\lambda_1 - 1} - \Sigma_{i=1}^{T} \lambda_1^{-i} . W_{t+i} \right)$$

Hence, to generate a consumption series, we choose a λ_2 value, which given a value of n determines the values of b and λ_1. Equation (3.16) can then be used to start from an initial condition C_{-1} to get a consumption series. If we assume that W_t is roughly constant (and equal to W^*) on the time interval $[t+T, \infty)$, the approximate error due to truncation at (3.16) is then given by:

$$\frac{W^*}{\beta . b} . \frac{1}{\lambda_1^T . (\lambda_1 - 1)}$$

This error will clearly be smaller the bigger the absolute value of λ_1 and the larger the number T of terms included in the approximation. Finally, we get a time series for output from (3.4), and a realization for ϵ_t from (3.3). It is therefore clear that the autoregression (3.8) does not introduce a new shock ν_t into the model, for ϵ_t and ν_t are one an exact function of the other. However, the nonlinear nature of this relationship implies that stochastic assumptions made on ν_t will not translate into similar properties for ϵ_t. In particular we will show that even when ν_t is independently and identically distributed over time, the technology shock ϵ_t may be autocorrelated.

Nonuniqueness of the solution:

Linear rational expectations models are characterized by not having, in general, a unique solution. The analytically more complex nonlinear models are not different in that respect. An adavantage of our solution method is to clearly point out the source of nonuniqueness in the model. For the one in this paper, we have just shown that any stationary autoregression assumed for the process Z_t will produce a stationary, and hence acceptable, solution.

IV. EMPIRICAL RESULTS.

The deterministic steady state of the model provides some guidance in choosing parameter values for the simulations. Those values are chosen so that the implied steadys state values of the variables are close to the ones in actual detrended time series data. In general, it is not possible to match the steady state values for all the variables, for the model imposes restrictions among those values which do not hold in actual data. For example, the budget constraint (2.5) implies that the steady stat values of consumption and output are the same. This observation, while being important because it introduces a constraint the parameter values to be used when solving the model is not however a result that emerges from actual time series data.

To obtain the deterministic steady state of the economy we set the values of the random shocks equal to their unconditional mean (zero) at all times t, to get:

$$(Z^*,K^*,W^*,C^*,r^*,Y^*) = \left(\frac{\mu}{1-A};\ \alpha+\frac{1-\beta^{-1}}{\theta};\ \frac{\mu}{1-A};\ \frac{1}{n}\left(1-\frac{\mu}{1-A}\right);\ \beta^{-1}-1;\ \frac{(1-\beta^{-1})^2}{2\theta}+\gamma+\alpha \right).$$

which as we can see, is independent of the values of λ_1 and λ_2.

The monthly discount factor was chosen $\beta = .997$, which implies annual interest rate values fluctuating around 3.6 %. The parameters in the Z_t-autoregression were chosen $A = .90$, $\mu = .03$, and $\sigma_\nu = 3\cdot10^{-4}$. Values of A close to 1.0 are plausible because one would expect Z_t to be a smooth process. Variations in the value of A in that range did not produce any noticeable change in the results. The value of μ just affects the mean values of the series but not the model's stochastic properties.

The values of n and α affect just the average consumption and capital stock values. They were chosen to be $n = .00165$ and $\alpha = 380.0$, which imply steady state values of consumption and the stock of capital equal to 424.24 and 374.0, respectively.

The value of ω was chosen to be 1.0 which implies that $\omega\cdot(K^*-\alpha) = -6.0$, well below -1.0, the necessary condition to guarantee stability of the solution. The value of θ affects the size of the fluctuations of the generated time series. A value of $\theta = .0005$ produced plausible size fluctuations. As a consequence of the restriction we mentioned above, the value of γ must be 47.24 . We chose the nonstationary root of $\Omega(L)$ to be $\lambda_1 = 1.20$, which implies values $\lambda_2 = .835$ and $b = .0289$. Experimentation with different values of λ_1 proved that choice not to produce any important change in the results. The value of ρ in the moving average representation (3.11) for η_t was chosen to be .25 , and the standard deviation of ξ_t was $\sigma_\xi = 10^{-4}$. While the value of ρ is mainly irrelevant for the results, the value of the variance is very important to obtain appropriate fluctuations in the solutions.

With these parameter values we generated monthly time series for (1960,1)-(1984,12). We then tried to replicate actual data collection by accumulating the flows of output and consumption over a quarter and averaging the interest rate values. These aggregated/averaged series were used in all the computations. A total of 50 simulations were run and we calculated the sample means and standard deviations for all the statistics we obtained. All the results we present are these sample means, whereas the numbers in brackets are the standard deviations obtained from the empirical distribution. Since we omitted the first 10 quarters from all the computations to minimize sampling error, a total of 90 observations were available. A number of them had to be skipped in each case depending on the number of lags envolved in the calculus.

To characterize the stochastic properties of the univariate time series produced by the model we computed the autocorrelation and partial autocorrelation functions for output, interest rates and consumption, which can be seen in table 1 . There is evidence in these functions that consumption and output could each be represented by an AR(2) model, and interest rates by an ARIMA(0,1,1) model. These short autoregressions for output and consumption match actual data well. Another interesting observation is that the simulated technology shock shows important serial correlation, and seems to be well represented by an AR(1) model with coefficient .94 . Autocorrelation in the technology shock is a major reason to produce the serial correlation in output that we observe in table 1 as well as in actual output data.

In table 2 we see that there is a weak positive effect from interest rates lagged 1 to 4 quarters to output, as well as a weak negative effect from output to interest rates 1 to 3 quarters ahead. These series are contemporaneously uncorrelated. The contemporaneous correlation between consumption and output is very important and extends up to 3 quarters in each direction. This property matches a characteristic of actual time series data, except that in actual data the correlation extends to a longer period. It is not surprising after these two observations to see that the correlations between interest rates and consumption have the same characteristics as those between interest rates and output.

The impulse responses of bivariate systems to shocks in one variable are presented in table 3 . Output has a smooth negative response to an interest rate innovation, recovering after more than two years. This is the type of response of output to nominal rates in actual data. The interest rates in our model are real, so these two similar results would just be comparable with no price uncertainty.

The response of consumption to an interest rate innovation is positive for two quarters and then becomes negative until it recovers to zero. The responses of interest rates to output or consumption innovations are initially positive to then become negative. In either case they just last for about 6 quarters.

Remark 1.-The hessian of the utility function (2.3) is a tridiagonal matrix:

$$\begin{pmatrix} a_{11} & a_{12} & 0 & 0 & \cdots & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 & \cdots & 0 & 0 \\ 0 & a_{32} & a_{33} & a_{34} & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & a_{T-1,T} & a_{T,T} \end{pmatrix}$$

With:

$a_{ii} = \beta^{i}.B$, $B = -[n+b\cdot(1+\beta)]$, $i=0,\dots,T$

$a_{i,i+1} = a_{i+1,i} = \beta^{i+1}.b$

This matrix can be decomposed (see Johnson and Riess [3]) as the product L.U of the matrices:

$$L = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ x_{21} & 1 & 0 & \dots & 0 \\ 0 & x_{32} & 1 & \dots & 0 \\ \dots & & & & \dots \\ 0 & 0 & 0 & x_{T-1,T} & 1 \end{pmatrix} \quad U = \begin{pmatrix} u_{11} & a_{12} & 0 & 0 & \dots & 0 \\ 0 & u_{22} & a_{23} & 0 & \dots & 0 \\ \dots & & & & & \dots \\ \dots & & & & & a_{T-1,T} \\ 0 & 0 & & & & u_{T,T} \end{pmatrix}$$

so that: $|H| = |L|.|U| = |U| = \Pi_1^T u_{ii}$

Under our parameter choice, the values of the u_{ii} elements alternate in sign starting with negative, and consequently, for these parameter values, the lifetime utility (2.5) is a concave function.

Remark 2.- Equations (3.5) and (3.9) imply the forecasting formula:

(3.17) $$E_t\left(\frac{Z_{t+2}}{1+\omega.\Delta K_{t+1}}\right) = \frac{A.(\beta^{-1}-A).Z_t+\mu.(\beta^{-1}-A-1)}{-\theta.(K_t-\alpha)}$$

for a nonlinear function of Z_{t+2} and K_{t+1}, given K_t. That conditional expectation is a nonlinear function of Z_t and K_t. However, that formula does not help us to identify any particular linear stochastic structure for $Z_{t+2}/(1+\omega\cdot K_{t+1})$ and hence, does not produce any inconsistency with the set of assumptions made above. In particular, it is consistent with the first order autoregression for Z_t assumed in (3.8).

Remark 3.- An alternative estimation strategy would start by postulating a finite order autoregression for the marginal utility of comsumption. For simplicity, we assume a first order autoregression:

(3.18) $W_{t+1} = B.W_t + b + \varepsilon_{t+1}$

Along a stationary equilibrium path, consumption will fluctuate around its steady state and consequentely, the discounted marginal utility of consumption would go to zero. In order not to impose any restriction on the parameters of (3.18), we chose to make our assumption on the undiscounted, rather than the discounted marginal utility. Then, from (3.7):

$$E_t(Z_{t+2}-\beta^{-1}\cdot Z_{t+1}) = \theta\cdot(K_t-\alpha)\cdot(BW_t+b)$$

which suggests the specification:

$$Z_{t+1} = \beta^{-1}\cdot Z_t + \theta\cdot(K_{t-1}-\alpha)\cdot(BW_{t-1}+b) + \xi_{t+1}$$

with $E_{t-1}\xi_{t+1}=0$ but $E_t\xi_{t+1}$ may be different from zero. If we now use our definition of the process Z_t:

$$Z_{t+1} = W_t\cdot[1+w(K_t-K_{t-1})]$$

then we get:

$$W_t[1+w(K_t-K_{t-1})] = \beta^{-1}W_{t-1}[1+w(K_{t-1}-K_{t-2})] + \theta(K_{t-1}-\alpha)(BW_{t-1}+b) + \xi_{t+1},$$

or,

$$K_t(wW_t)-\left[wW_t+\beta^{-1}wW_{t-1}+\theta(BW_{t-1}+b)\right]K_{t-1}+\beta^{-1}W_{t-1}K_{t-2}+[W_t-\beta^{-1}W_{t-1}+\theta\alpha(BW_{t-1}+b)]-\xi_{t+1} = 0$$

which is a non linear second order difference equation on (W_t, K_t) with nonlinear coefficients, which makes impossible to analyze the equilibrium of the model in a way like the one we suggested in section 3 .

V. CONCLUSIONS

We have analyzed in this paper the role that costs of adjustment of the optimal capital stock play on the equilibrium determination of interest rates. Considering endogenous rates of interest in optimal capital accumulation models creates difficulties, since the implied decision rules are nonlinear. Under uncertainty, these decision rules will also include expectations of future variables, and that combination of things makes impossible the use of standar methods for the analysis of dynamic economic models.

A general method that has recently been introduced is utilized to solve the nonliner rational expectations model in this paper. The method allows for generation of equilibrium time series data that can be used to characterize the model's properties concerning the interrelations between output, consumption, capital and interest rates. It is shown in the paper that our simple general equilibrium model of capital accumulation is able to explain some autocorrelation properties as well as interesting cross-correlations that are observed in actual time series data.

TABLE 1

I. AUTOCORRELATION FUNCTIONS.

	K = 1	2	3	4	5	6	7	8	9	10
Y_t	.87	.61	.31	.49	-.13	-.24	-.29	-.30	-.30	-.30
	(.02)	(.06)	(.10)	(.11)	(.10)	(.08)	(.11)	(.14)	(.17)	(.19)
ΔY_t	.71	.23	-.09	-.25	-.30	-.27	-.20	-.13	-.07	-.05
	(.04)	(.11)	(.16)	(.17)	(.14)	(.11)	(.13)	(.17)	(.16)	(.14)
$\Delta^2 Y_t$	.32	-.25	-.25	-.20	-.15	-.07	.0	.17	.02	.07
	(.07)	(.09)	(.08)	(.13)	(.11)	(.06)	(.06)	(.12)	(.10)	(.08)
r_t	-.26	-.19	.0	.07	-.10	-.03	.04	-.03	.0	.0
	(.07)	(.09)	(.03)	(.10)	(.08)	(.04)	(.08)	(.12)	(.07)	(.07)
Δr_t	-.52	-.05	.05	.10	-.09	.0	.05	-.03	.02	-.03
	(.06)	(.09)	(.05)	(.10)	(.10)	(.04)	(.10)	(.13)	(.09)	(.07)
$\Delta^2 r_t$	-.65	.12	.01	.08	-.08	.0	.05	-.04	.04	-.04
	(.04)	(.08)	(.08)	(.11)	(.11)	(.06)	(.11)	(.14)	(.11)	(.08)
C_t	.88	.61	.31	.05	-.13	-.24	-.29	-.30	-.30	-.30
	(.02)	(.06)	(.10)	(.11)	(.10)	(.08)	(.11)	(.14)	(.17)	(.19)
ΔC_t	.72	.24	-.09	-.26	-.30	-.27	-.20	-.13	-.08	-.05
	(.04)	(.11)	(.16)	(.17)	(.14)	(.11)	(.14)	(.18)	(.16)	(.15)
$\Delta^2 C_t$	.35	-.25	-.27	-.21	-.16	-.07	.0	-.02	.03	.07
	(.07)	(.10)	(.08)	(.13)	(.12)	(.06)	(.07)	(.12)	(.10)	(.08)
ϵ_t	.94	.88	.80	.72	.63	.54	.45	.36	.28	.20
	(.0)	(.02)	(.03)	(.04)	(.06)	(.07)	(.08)	(.09)	(.10)	(.10)
$\Delta \epsilon_t$	.89	.75	.59	.45	.31	.20	.10	.02	-.04	-.08
	(.02)	(.04)	(.06)	(.08)	(.09)	(.10)	(.11)	(.11)	(.11)	(.11)
$\Delta^2 \epsilon_t$	.38	.22	.09	.02	-.04	-.06	-.09	-.08	-.07	-.07
	(.01)	(.05)	(.03)	(.03)	(.03)	(.05)	(.06)	(.05)	(.04)	(.06)

II. PARTIAL AUTOCORRELATION FUNCTIONS.

K =	1	2	3	4	5	6	7	8	9	10
Y_t	.87	-.67	.10	-.10	-.14	-.08	-.04	-.11	-.08	-.04
	(.02)	(.10)	(.19)	(.07)	(.10)	(.07)	(.07)	(.03)	(.08)	(.08)
ΔY_t	.71	-.55	.10	-.22	-.03	-.12	-.05	-.09	-.05	-.05
	(.04)	(.08)	(.09)	(.06)	(.09)	(.04)	(.07)	(.03)	(.11)	(.14)
$\Delta^2 Y_t$	.32	-.41	-.03	-.25	-.11	-.15	.-11	-.13	-.12	-.03
	(.07)	(.04)	(.07)	(.08)	(.06)	(.08)	(.02)	(.07)	(.15)	(.01)
r_t	-.26	-.29	-.17	-.05	-.15	-.12	-.07	-.12	-.09	-.11
	(.07)	(.07)	(.07)	(.07)	(.05)	(.08)	(.14)	(.03)	(.07)	(.09)
Δr_t	-.52	-.45	-.39	-.20	-.19	-.20	-.13	-.15	-.11	-.15
	(.06)	(.03)	(.06)	(.07)	(.01)	(.07)	(.12)	(.07)	(.08)	(.08)
$\Delta^2 r_t$	-.65	-.52	-.48	-.30	-.18	-.16	-.06	-.07	.0	-.01
	(.04)	(.03)	(.06)	(.06)	(.08)	(.07)	(.11)	(.11)	(.06)	(.05)
C_t	.88	-.67	.10	-.10	-.01	-.08	-.05	-.11	-.08	-.04
	(.02)	(.11)	(.21)	(.08)	(.10)	(.07)	(.07)	(.03)	(.08)	(.07)
ΔC_t	.72	-.57	.13	-.23	-.01	-.13	-.04	-.09	-.05	-.05
	(.04)	(.08)	(.09)	(.06)	(.09)	(.05)	(.07)	(.03)	(.10)	(.14)
$\Delta^2 C_t$	.35	-.43	.0	-.26	-.10	-.16	-.11	-.14	-.11	-.34
	(.07)	(.04)	(.07)	(.08)	(.06)	(.08)	(.01)	(.06)	(.15)	(.01)
ϵ_t	.94	-.13	-.11	-.09	-.08	-.06	-.05	-.04	-.03	-.04
	(.0)	(.03)	(.02)	(.01)	(.02)	(.01)	(.02)	(.02)	(.02)	(.01)
$\Delta \epsilon_t$	.89	-.22	-.12	-.06	-.03	-.05	-.02	-.04	-.01	.0
	(.02)	(.03)	(.04)	(.05)	(.02)	(.03)	(.05)	(.03)	(.02)	(.04)
$\Delta^2 \epsilon_t$	.38	.08	-.01	-.04	-.04	-.03	-.06	-.02	-.02	-.03
	(.01)	(.07)	(.04)	(.03)	(.03)	(.04)	(.04)	(.01)	(.04)	(.05)

TABLE 2

III. CROSS-CORRELATION FUNCTIONS.

K	$Y_t - r_t$	$Y_t - C_t$	$r_t - C_t$
-8	.04 (.07)	-.33 (.13)	-.02 (.04)
-7	.03 (.06)	-.33 (.14)	.0 (.04)
-6	.0 (.03)	-.28 (.14)	.05 (.04)
-5	-.04 (.05)	-.16 (.14)	.10 (.03)
-4	-.09 (.06)	.04 (.13)	.15 (.04)
-3	-.16 (.07)	.31 (.11)	.17 (.05)
-2	-.22 (.06)	.62 (.06)	.20 (.05)
-1	-.23 (.06)	.88 (.02)	.17 (.03)
0	-.05 (.03)	1.0 (.0)	-.05 (.03)
1	.17 (.03)	.88 (.02)	-.22 (.06)
2	.20 (.05)	.62 (.06)	-.22 (.06)
3	.17 (.05)	.31 (.11)	-.16 (.07)
4	.15 (.04)	.04 (.13)	-.09 (.06)
5	.09 (.03)	-.16 (.14)	-.04 (.05)
6	.05 (.04)	-.28 (.14)	.0 (.03)
7	.0 (.04)	-.33 (.14)	.03 (.06)
8	-.02 (.04)	-.33 (.13)	.04 (.07)

TABLE 3

IMPULSE RESPONSES

I. System: Output - Interest Rates.

Quarters ahead	Output Responses		Interest Rates Responses	
	Shock in Output	Shock in Interest Rates	Shock in Output	Shock in Interest Rates
1	1.0 (.0)	.0 (.0)	.49 (.08)	.87 (.04)
2	2.09 (.12)	-.40 (.13)	.06 (.06)	-1.09 (.13)
3	2.62 (.33)	-1.47 (.33)	.10 (.08)	-.16 (.16)
4	2.49 (.53)	-1.79 (.44)	-.29 (.11)	.37 (.11)
5	1.82 (.67)	-1.38 (.50)	-.18 (.10)	.19 (.12)
6	1.08 (.69)	-.87 (.47)	-.05 (.05)	-.02 (.14)
7	.42 (.58)	-.50 (.40)	-.07 (.06)	-.02 (.07)
8	-.13 (.39)	-.16 (.30)	-.07 (.06)	.10 (.08)
12	-.76 (.52)	.50 (.30)	.02 (.03)	.0 (.02)
16	.0 (.29)	.07 (.22)	.02 (.02)	-.01 (.05)
20	.30 (.22)	-.18 (.10)	.0 (.01)	.0 (.01)
24	.04 (.22)	-.05 (.15)	.0 (.0)	.0 (.0)

IMPULSE RESPONSES

II. System: Interest Rates - Consumption.

Quarters ahead	Interest Rates Responses		Consumption Responses	
	Shock in Interest Rates	Shock in Consumption	Shock in Interest Rates	Shock in Consumption
1	1.0 (.0)	.0 (.0)	.47 (.08)	.88 (.04)
2	-.95 (.13)	.58 (.06)	.61 (.22)	2.08 (.18)
3	-.09 (.12)	.19 (.12)	-.10 (.43)	3.14 (.42)
4	.19 (.07)	-.42 (.13)	-.45 (.50)	3.21 (.63)
5	.08 (.11)	-.26 (.13)	-.41 (.47)	2.42 (.79)
6	-.05 (.12)	-.05 (.10)	-.31 (.34)	1.47 (.84)
7	-.04 (.05)	-.04 (.07)	-.28 (.21)	.66 (.72)
8	.06 (.07)	-.10 (.08)	-.21 (.11)	-.02 (.52)
12	.0 (.0)	.02 (.04)	.11 (.17)	-.97 (.61)
16	.0 (.0)	.02 (.02)	.07 (.09)	-.06 (.37)
20	.0 (.0)	.0 (.02)	-.03 (.06)	.38 (.25)
24	.0 (.0)	-.01 (.01)	-.04 (.06)	.08 (.28)

TABLE 4

AUTOREGRESSIVE SYSTEMS

I. Output - Interest Rates.

	Y_{t-1}	Y_{t-2}	Y_{t-3}	Y^{t-4}	r_{t-1}	r_{t-2}	r_{t-3}	r_{t-4}	Const.	R^2	Q	SEE
Y_t	2.31 (.16)	-1.62 (.28)	.15 (.18)	.06 (.06)	-.02 (.0)	-.06 (.0)	-.02 (.0)	.0 (.0)	43.5 (11.3)			
r_t	14.1 (1.85)	-11.0 (4.7)	-7.9 (5.2)	4.71 (2.0)	-1.25 (.12)	-1.46 (.11)	-.74 (.10)	-.14 (.07)	81.1 (204.5)	.62 (.04)	27.3 (8.7)	

$$\Sigma = \begin{pmatrix} \underset{(.0007)}{.0039} & \underset{(.01)}{.04} \\ \\ \underset{(.08)}{.49} & \underset{(.31)}{1.72} \end{pmatrix} \qquad \ln |\Sigma| = \underset{(.30)}{-5.3}$$

II. Interest Rates - Consumption.

	r_{t-1}	r_{t-2}	r_{t-3}	r_{t-4}	C_{t-1}	C_{t-2}	C_{t-3}	C_{t-4}	Const.	R_2	Q	SEE
r_t	-1.26 (.12)	-1.45 (.11)	-.73 (.11)	-.14 (.07)	14.63 (1.91)	-11.8 (5.09)	-7.74 (5.6)	4.83 (2.14)	49.4 (196.7)			
C_t	-.02 (.0)	-.05 (.0)	-.02 (.0)	.0 (.0)	2.38 (.16)	-1.78 (.29)	.28 (.18)	.03 (.06)	41.0 (10.5)	.97 (.01)	21.7 (7.8)	

$$\Sigma = \begin{pmatrix} \underset{(.31)}{1.69} & \underset{(.009)}{.036} \\ \\ \underset{(.08)}{.47} & \underset{(.0006)}{.0034} \end{pmatrix} \qquad \ln |\Sigma| = \underset{(.30)}{-5.45}$$

APPENDIX 1.

Consider the consumer's optimization problem:

$$\max_{\{C_t,K_t\}} E_0 V(\{C_s\}_{s=-1}^{\infty}) = E_0 \sum_{t=0}^{\infty} \beta^t . U(C_t) = E_0 \sum_{t=0}^{\infty} \beta^t . (C_t - \frac{n}{2} \cdot C_t^2 - \frac{b}{2} \cdot (C_t - C_{t-1})^2)$$

subject to the constraints:

$$C_t + K_t - K_{t-1} + w/2 . (K_t - K_{t-1})^2 = Y_t$$

$$Y_t = \gamma - \frac{\theta}{2} \cdot (K_{t-1} - \alpha)^2 + \epsilon_t$$

$$C_t ,\ K_t \geq 0$$

Following the approach in Kushner [..], there exists a sequence of random Lagrange multipliers $\{\lambda_t\}$ such that:

$$E_t V_{C_t} - E_t \lambda_t \leq 0 \quad \text{and} \quad = 0 \quad \text{if} \quad C_t > 0$$

$$-E_t \lambda_t . [1 + w . \Delta K_t] + E_t \{[-\theta(K_t - \alpha) + w(K_{t+1} - K_t) + 1] \lambda_{t+1}\} \leq 0 \quad \text{and} \quad = 0 \quad \text{if} \quad K_t > 0$$

and the transversality condition:

$$\lim_{T} \beta^t \cdot K_T \cdot \lambda_T = 0$$

where E_t denotes the operator expectation conditional on the sigma algebra $\mathcal{F}_t$ of subsets of Ω_t, a set which includes current and past decision and states. Hence, the conditional expectation of any current decision variable is that same variable.

Assuming an interior solution, i.e., $C_t, K_t > 0$ $\forall t$ and using standard properties of the conditional expectation operator, we get:

$$E_t V_{C_t} = E_t \lambda_t$$

and:

$$E_{t+1} V_{C_{t+1}} = E_{t+1} \lambda_{t+1} \Rightarrow E_t V_{C_{t+1}} = E_t \lambda_{t+1}$$

(1) $$E_t V_{C_t} . (1 + w . \Delta K_t) = E_t \{\lambda_{t+1} . [1 - \theta(K_t - \alpha) + w . \Delta K_{t+1}]\} =$$

$$E_t \left(E_{t+1} \{\lambda_{t+1} [1 - \theta(K_t - \alpha) + w . \Delta K_{t+1}]\} \right) = E_t \left([1 - \theta(K_t - \alpha) + w . \Delta K_{t+1}] . E_{t+1} \lambda_{t+1} \right) =$$

$$= E_t \left([1 - \theta(K_t - \alpha) + w . \Delta K_{t+1}] . E_{t+1} V_{C_{t+1}} \right) = E_t \left(E_{t+1} \{[1 - \theta(K_t - \alpha) + w . \Delta K_{t+1}] . V_{C_{t+1}}\} \right) =$$

$$= E_t \{[1 - \theta(K_t - \alpha) + w . \Delta K_{t+1}] . V_{C_{t+1}}\}$$

APPENDIX 2.- In order to clarify the equilibrium relationships among real interest rates, the marginal rate of return on capital, and the marginal rate of time preference, we split now the optimization problem in appendix 1 into two problems, that of a representative consumer who lends to the single firm in the economy, consuming each period his endowment net of this loans, and the problem of a firm which takes care of production, borrows from consumers (that is, issues some one period bonds B_t which are sold to consumers), and distributes some dividends Y_t^* to the owners, in this case, the representative consumer in the economy.

a)

Consider now the optimization problem of a consumer whose only possibility of transfering resources over time is by buying bonds (lending) from the production firm in the economy:

$$\underset{\{C_t,\ B_t\}^{\infty}_{t=0}}{\text{Max}} \quad E_0V = E_0 \sum^{\infty}_{t=0} \beta^t.(C_t - \frac{n}{2}\cdot\gamma\ C_t{}^2 - \frac{b}{2}\cdot(C_t - C_{t-1})^2)$$

subject to:

$$C_t + B_t = Y^*{}_t + (1+r_{t-1})B_{t-1}$$

given r_{-1}, C_{-1} and B_{-1} and taking the sequences $\{r_t,\ Y^*{}_t\}$ as given.

The optimality conditions, as a function of the random Lagrange multipliers $\{\delta_t\}^{\infty}_{t=0}$ are:

$$E_tV_{C_t} - E_t\delta_t \leqq 0 \qquad \text{and} \quad = 0 \quad \text{if} \quad C_t < 0$$

$$E_t\delta_t - (1+r_t).E_t\delta_{t+1} \leqq 0 \qquad \text{and} \quad = 0 \quad \text{if} \quad B_t > 0$$

and the transversality condition:

$$\lim_T \delta_T.B_T = 0$$

which imposes a bound on the rate of growth of savings. But B_t is a flow variable, not a stock, and one would expect it to converge to a finite steady state value. If consumption behaves in a similar fashion, then the transversality condition will be satisfied.

Assuming an interior solution, then:

$$E_tV_{C_t} = E_t\delta_t \quad \text{and}$$

$$E_{t+1}V_{C_{t+1}} = E_{t+1}\delta_{t+1} \qquad \text{which implies:}$$

$$E_tV_{C_{t+1}} = E_t\delta_{t+1}$$

since:

$$E_t\delta_t = (1+r_t).E_t\delta_{t+1}, \qquad \text{then:}$$

(2) $$1+r_t = \frac{E_tV_{C_t}}{E_tV_{C_{t+1}}}$$

b)

If we now consider the optimization problem of a firm which maximizes the expected present value of the stream of dividends distributed among the owners each period:

$$\max_{\{Y^*_t,\, B_t,\, K_t\}^{\infty}_{t=0}} E_0 \sum^{\infty}_{t=0} D_t \cdot Y^*_t$$

where D_t is a generic sequence of discount factors used by the firm, and the problem is solved subject to the sequence of constraints:

$$Y_t + B_t = Y^*_t + (1+r_{t-1}) \cdot B_{t-1} + K_t - K_{t-1} + (\omega/2) \cdot (K_t - K_{t-1})^2$$

$$Y_t = \gamma - \frac{\theta}{2} \cdot (K_{t-1} - \alpha)^2 + \epsilon_t$$

and given K_{-1}, B_{-1}, r_{-1}.

Then, the following optimality conditions must hold at each time t :

$$E_t \eta_t - (1+r_t) \cdot E_t \eta_{t+1} \leq 0 \qquad \text{and } = 0 \text{ if } B_t > 0$$

$$E_t \left\{ [-\theta(K_t - \alpha) + w(K_{t+1} - K_t) + 1] \cdot \eta_{t+1} \right\} - E_t \left\{ [w(K_t - K_{t-1}) + 1] \eta_t \right\} \leq 0$$

$$\text{and } = 0 \text{ if } K_t > 0.$$

$$E_t D_t - E_t \eta_t \leq 0 \qquad \text{and } = 0 \text{ if } Y^*_t > 0$$

as well as the transversality condition:

$$\lim_{T} D_T K_T = 0$$

which imposes a bound on the rate of growth of the stock of capital.

Assuming an interior solution, i.e., Y^*_t, B_t, $K_t > 0$ $\forall t$, then:

(3) $$1 + r_t = \frac{E_t \eta_t}{E_t \eta_{t+1}} = \frac{E_t D_t}{E_t D_{t+1}}$$

(4) $$E_t \eta_t = E_t D_t$$

(5) $$E_t \left\{ \eta_{t+1} \cdot [1 - \theta(K_t - \alpha) + w \cdot \Delta K_{t+1}] \right\} = E_t \left\{ \eta_t [1 + w \cdot \Delta K_t] \right\}$$

Suppose that the firm uses as a (random) discount factor the marginal utility of the representative shareholder, $D_t = V_{C_t}$. Then (4) becomes identical to (2) and (5) becomes:

$$E_t \left\{ \eta_{t+1} [1 - \theta(K_t - \alpha) + w \cdot \Delta K_{t+1}] \right\} = (1 + w \cdot \Delta K_t) \cdot E_t \eta_t$$

From (4) we have that the difference between η_t and V_{C_t} is a random variable γ_{t+1} unpredictable as of time t :

$$\eta_t = V_{C_t} + \gamma_{t+1}$$

with $E_t \psi_{t+1} = 0$. We then have:

$$E_t\left\{\eta_{t+1}\cdot[1-\theta(K_t-\alpha)+\omega\cdot\Delta K_{t+1}]\right\} = E_t\left\{(V_{C_{t+1}}+\psi_{t+2})\cdot[1-\theta(K_t-\alpha)+\omega\cdot\Delta K_{t+1}]\right\} =$$

$$= E_t\left\{V_{C_{t+1}}\cdot[1-\theta(K_t-\alpha)+\omega\cdot\Delta K_{t+1}]\right\} + E_t\left\{\psi_{t+2}\cdot[1-\theta(K_t-\alpha)+\omega\cdot\Delta K_{t+1}]\right\}$$

and the second term in this sum is equal to:

$$E_t\left(E_{t+1}\left\{\psi_{t+2}\cdot[1-\theta(K_t-\alpha)+\omega\cdot\Delta K_{t+1}]\right\}\right) = E_t\left([1-\theta(K_t-\alpha)+\omega\cdot\Delta K_{t+1}]\cdot E_{t+1}\psi_{t+2}\right) = 0$$

Analogously,

$$E_t\left\{\eta_t\cdot[1+\omega\cdot\Delta K_t]\right\} = E_t\left\{V_{C_t}\cdot[1+\omega\cdot\Delta K_t] + E_t\left\{\psi_{t+1}\cdot[1+\omega\cdot\Delta K_t]\right\}\right. =$$

and the second term in this sum is equal to:

$$[1+\omega\cdot\Delta K_t].E_t\psi_{t+1} = 0$$

and therefore, equation (5) becomes identical to (1), the first order condition in the first of our optimization problems.

Actually, all that is needed to get this result is that the firm uses a discount factor D_t such that $E_t D_t = E_t V_{C_t}$. What this means is that the firm discounts by the current expected value of the marginal utility of current consumption.

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